Talks at the Mathematical Structures in Quantum Mechanics

1 Invited Speakers:

- Dariusz Chruściński: *Universal Constraint for Relaxation Rates for Quantum Dynamical Semigroup*

A general property of relaxation rates in open quantum systems is discussed. We find the constraint for relaxation rates that universally holds in fairly large classes of quantum dynamics, e.g., weak coupling regimes, as well as for entropy nondecreasing evolutions. We conjecture that this constraint is universal, i.e., it is valid for all quantum dynamical semigroups. The conjecture is supported by numerical analysis. Moreover, we show that the conjectured constraint is tight by providing a concrete model that saturates the bound. This universality marks an essential step toward the physical characterization of complete positivity as the constraint is directly verifiable in experiments. It provides, therefore, a physical manifestation of complete positivity.

- Franco Fagnola*: Entangled Stationary States of Gaussian Open Quantum Systems*

We consider 2-mode system with quadratic Hamiltonian in creation and annihilation operators in which each mode interacts with a reservoir. We show that the partial trace of on the 2-mode system of the stationary state of whole open system is entangled if the two reservoirs have different temperatures. (Based on a joint work with A. Dhahri, A. Joye and D. Poletti)

- Markus Grassl: *What do we know about the SIC-POVM problem?*

SIC-POVMs are generalised quantum measurements which are of particular interest in the context of quantum state tomography and quantum key distribution. Alternatively, they can be described by d^2 normalised vectors in the d-dimensional complex vector space such that the inner product between any pair of vectors has constant modulus.

It has been conjectured that SIC-POVMs exist for all dimensions and that they can be constructed as orbits of a so-called fiducial vector under the Weyl-Heisenberg group. Despite a lot of effort, numerical or exact fiducial vectors are still only known for a finite, though growing list of dimensions.

The talk will give a brief overview of the current state of the conjecture. It will be discussed how observations based on existing solutions allow us to find more solutions, leading to further insights and conjectures.

The talk includes recent results obtained in collaboration with Marcus Appleby, Ingemar Bengtsson, Michael Harrison, and Gary McConnell.

- Thomas Schulte-Herbrueggen: *The Role of Markovianity in Quantum Thermodynamics*

We connect quantum control theory with quantum thermodynamics within the framework of Lie-semigroup theory. In particular, we sketch how to construct the Markovian counterparts of several types of quantum Thermal Operations via their respective Lie wedge.

In an explicit qubit example, we parameterise the Markovian subset of maps within the set of all the Thermal Operations.

As an application, we give inclusions in terms of d-majorisation for reachable sets of bilinear control systems, where coherent controls are complemented by switchable couplings to a thermal bath as additional resource.

Joint work with Frederik vom Ende, Emanuel Malvetti, and Gunther Dirr based on arXiv:2303.01891 and Open Syst. Inf. Dyn. 30 (2023), 2350005.

- Jerzy Juliusz Kijowski: *Quanta versus Geometry*

The textbook version of quantum mechanics is incompatible with geometry: Cartesian coordinates are preferred, and transformation to curvilinear coordinates often leads to inconsistencies. In this talk, I will show the strength - but also the limitations - of the geometric approach to quantization. The existing discrepancy between quanta and curved space-time will be discussed.

- Seung-Hyeok Kye: *An unified approach to various criteria through ampliation arising from quantum information theory*

Abstract: We begin with the elementary positive maps Ad_s between matrix algebras which sends x to s*xs, to explore various kinds of positive maps. We use the duality to define k-positivity and completely positivity. Using Choi matrices, we see that separable states corresponds to the maps Ad_s with rank of s = 1. A bi-partite state which not separable is called entanglement, which is now considered as one of the most important resources in the current information theory. We exhibit an identity to connect compositions and tensor products of linear maps between matrix algebras through Choi matrices. With this identity, various known criteria for the above notions through ampliation can be obtained in a single framework. Through the discussion, the role of Choi matrices is crucial, and we see that the definition of Choi matrices with a specific basis does not depend on the choices of bases, but depends on the bilinear pairings given by bases. For this purpose, we examine the definition of Choi matrices in the level of vector spaces. This talk is based on the following papers: Choi matrices revisited, J.Math. Phys. 63 (2022), 092202 - Compositions and tensor products of linear maps between matrix algebras, Linear Algebra Appl. 658 (2023), 283-309. - in preparation (with Kyung-Hoon Han)

- Marcin Markiewicz: *Averaging physical quantities over non-compact symmetry groups: t-designs and duality of averaging.*

Averaging physical quantities over Lie groups appears in many contexts across the rapidly developing branches of physics like quantum information science or quantum optics. Such an averaging process can be always represented as averaging with respect to a finite number of elements of the group, called a finite averaging set. In the previous research such sets, known as t-designs, were constructed only for the case of averaging over unitary groups (hence the name unitary t-designs). It is also a well-established fact, that uniform averaging over the collective action of a unitary group on a multipartite quantum state projects the state to a form equivalent to a permutation operator of the subsystems. Hence states equivalent to permutation operators are untouched by collective unitary noise. A trivial observation shows that uniform averaging over permutation operators projects the state into a form with block-diagonal structure equivalent to the one of the collective action of the unitary group. We introduce a name for this property: duality of averaging. In this talk we investigate the problem of constructing finite averaging sets and duality of averaging for averaging over general non-compact matrix Lie groups. We provide a general construction of finite averaging sets based on the Cartan decomposition of the group, which splits the group into its compact and non-compact components [1]. We also show, that in the case of finite dimensional quantum systems duality of averaging holds for any pairs of symmetry groups being dual reductive pairs, regardless of whether they are compact or not, as long as the averaging operation is defined via iterated integral over the Cartan decomposition of the group action [2]. Although our results are very general, we focus much attention on the concrete example of averaging multipartite quantum states over non-unitary SLOCC-type (Stochastic Local Operations and Classical Communication) operations.

References: [1]. M. Markiewicz and J. Przewocki (2021), J. Phys. A: Math. Theor. 54, 235302. [2]. M. Markiewicz and J. Przewocki (2022), arXiv:2208.07689v2 [quant-ph] .

- Mate Matolcsi: *MUB-triplets and Hadamard cubes*

It is well-known that a pair of mutually unbiased bases (MUBs) $X = \{x_1, \dots x_n\}, Y = \{y_1, \dots, y_n\}$ gives rise to a complex Hadamard matrix with entries $h_{j,k} = \langle x_j, y_k \rangle$. In a similar fashion, a triplet of MUB's X, Y, Z gives rise to a 3-dimensional object that we call a Hadamard cube: $c_{j,k,r} = \langle x_j, y_k \rangle \langle y_k, z_r \rangle \langle z_r, x_j \rangle$. We establish a list (P) of basic properties of such a Hadamard cube induced by a MUB-triplet. Vice versa, we show that any cube possessing properties (P) determines a MUB-triplet up to unitary equivalence. Finally, we present some numerical data of Hadamard cubes in dimension 6. This gives a clue to a possible proof of the conjecture that the maximal number of MUB's in dimension 6 is 3. Joint work with A. Matszangosz, D. Varga and M. Weiner.

- Hiroyuki Osaka: *Factorization property of positive maps on C*-algebras and its application*

The purpose of this talk is to present a factorization property of positive linear maps on C*-algebras : Let A, C be C*-algebras and let H be a Hilbert space. Suppose that $\alpha: A \to C$ is a linear map and $\beta: A \otimes B(H) \to C \otimes B(H)$ is a positive linear map and $\alpha \otimes id - \beta$ is positive. Then $\beta = \gamma \otimes id$ for some positive linear map $\gamma: A \to C$. As an application we give a reciep of construction of indecomposable positive maps on C*-algebras which is extention of Piani and Mora (revisited by Huber et al. in 2018). This method is useful to detect quantum entanglument in Quantum Information Theory. We also discuss about k-entanglument witness. This is mainly based on joint work with B. V. Rajarama Bhat (Int. J. Quantum Inf. 18 (2020), no. 5, 2050019) and Tomasz Mlynik and Marcin Marciniak (arXiv:2104.14058 ver4., 2022).

- Francisco Javier García Pacheco: *Quantum systems via C*-algebras and effect algebras.*

According to the First Postulate of Quantum Mechanics, a quantum mechanical system is represented by an infinite-dimensional separable complex Hilbert space. The observable magnitudes are represented by the (bounded or unbounded) selfadjoint operators. The unsharp measurements (also called effects) are represented by the positive selfadjoint bounded operators lying below the identity. The quantum bits are the unit vectors of the Hilbert space and the quantum states are the complex rays generated by the quantum bits. There are two main universal algebras involved in this frame: C*-algebras and effect algebras. C*-algebras model the behavior of observable measurements. Effect algebras model the behavior of unsharp measurements. We will present the most recent developments on this trend

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- Fernando Leon Saavedra: *Operators with minimal commutants and the double commutant propety*

An linear operator T defined on a separable Hilbert space H has a minimal commutant, if its commutant is the small as possible. That is, it is the WOT clousure of the algebra generated by T and the identity. And T has the double commutant property if its bicommutant is also the WOT clousure of the algebra generated by T and the identity. In this lecture we will focused on such properties for composition operators defined on the Hardy space $H^2(D)$.

- Karol Życzkowski: *Energy Distance between pure quantum states and Quantum Transport Problem*

It is often important to distinguish between two given quantum states or to describe how close they are apart. Depending on the envisaged application, various distances are employed. While the standard solutions (including trace distance D_{Tr}, Hilbert-Schmidt distance or the fidelity-based, Bures distance) are unitarily invariant, for some purposes it is convenient to work with more general notions.

Basing on the transport problem of Monge-Kantorovich-Wasserstein and their quantum analogues [1,2] we introduce an *energy distance* W_H determined by any Hamiltonian H. For eigenstates $|\psi_i\rangle$ of the Hamiltonian, $\langle\psi_i|H|\psi_i\rangle = E_i$, their distance is given by the absolute value of the energy difference, $W_H(|\psi_i\rangle, |\psi_j\rangle) = |E_i - E_j|$. Thus for the hydrogen atom, the distance from the ground state $|0\rangle$ to a Rydberg state is much larger than to the first excited state, $W_H(|0\rangle, |100\rangle) \gg W_H(|0\rangle, |1\rangle)$, although the trace distance and distinguishability between all three orthogonal states are equal, $D_{\mathrm{Tr}}(|0\rangle, |100\rangle) = D_{\mathrm{Tr}}(|0\rangle, |1\rangle) = D_{\mathrm{Tr}}(|1\rangle, |100\rangle)$. The energy distance is analyzed for several physical models. A quantum speed limit relation is established: the time T required to transform a state $|\psi\rangle$ into $|\phi\rangle$ by any interaction Hamiltonian V is bounded from below by the ratio,

[1] K. Życzkowski and W. Słomczyński, Monge distance between quantum states, *J. Phys.* **A 31**, 9095 (1998).

[2] S. Friedland, M. Eckstein, S. Cole, K. Życzkowski, Quantum Monge-Kantorovich problem and transport distance between density matrices, *Phys. Rev. Lett.* **129**, 110402 (2022).

* joint work with Rafał Bistroń, Michał Eckstein, Shmuel Friedland and Tomasz Miller

2 Other speakers:

- Remigisz Augusiak: *Entangled subspaces and their characterization*

Genuinely entangled subspaces are a class of subspaces in the multipartite Hilbert spaces that are composed of only genuinely entangled states and are thus a natural generalization of the completely entangled subspaces to the multipartite regime. They are thus an interesting object of study in the context of multipartite entanglement. In this talk I will present some of our recent results concerning characterization of this type of subspaces (such as criteria for being genuinely entangled or self-testing statements), putting a particular emphasis on the stabilizer subspaces known in the context of quantum error stabilizer codes. In particular, I will discuss a conjecture that there are no genuinely entangled multipartite stabilizer states (mixed states defined on stabilizer subspaces) with positive partial transpositions. This talk is based on a series of papers [1, 2, 3, 4]. [1] M. Demianowicz and R. Augusiak, Phys. Rev. A 100, 062318 (2019). [2] F. Baccari, R. Augusiak, I. ˇSupi´c, and A. Ac´ın, Phys. Rev.Lett. 125, 260507 (2020). [3] O. Makuta, R. Augusiak, New J. Phys. 23, 043042 (2021). [4] O. Makuta, B. Kuzaka, R. Augusiak, Quantum 7, 915 (2023).

- Michał Banacki: *Remarks on steering in a C*-algebraic framework*

We discuss a scenario of bipartite steering with local subsystems of the parties modeled by certain operator algebras. In particular, we formalize the notion of quantum assemblages in a commuting observables paradigm and focus on equivalent descriptions of such objects. Finally, we provide some no-go results concerning this class of assemblages.

- Anindita Bera: *New classes of optimal entanglement witnesses*

During my presentation, I will be discussing new classes in entanglement witnesses (EWs) and their optimality. These classes of witnesses generalise the known families of entanglement witnesses, mainly the Choi and the reduction. I will talk about a nontrivial optimization procedure, which is needed if the witnesses are not optimal. While we can use the spanning criterion to achieve optimality, it may not be enough. Some entanglement witnesses are proven to be optimal, without satisfying the spanning criterion.

- Michał Kaczor: *Time evolution of entanglement in tetrahedral structure of spins located in Markovian environment*

Quantum entanglement has gained a lot of scientifical interest in recent years as one of the most important resources in Quantum Informatics. However, as valueable as it is, it is also incredibly sensitive on the external factors. For this reason, when designing systems that are to be used later in quantum information protocols, it is important to specify parameters of the applied model and the external conditions to preserve the entanglement for the time required to perform the computation. Moreover, the layouts intended to be applied in Quantum Informatics has to be analyzed as an open system - the interaction with the environment is unavoidable; however, certain assumptions (i.e. Markovian dynamics of the system) can make the analysis efficiently simplified. The system of interest in this presentation is a tetrahedron of spins $\mathrm{s} = \frac{1}{2}$ with (or without) additional spin in the centre - such structures can be found in crystallic materials, similar to quartz structure. The comparison of three cases was made (without the central spin, with central spin $\mathrm{s} = \frac{1}{2}$ or $\mathrm{S} = 1$) in terms of entanglement time evolution for this system in Markovian environment. Various parameters of the model were taken into account - exchange integrals, Dzyaloshinskii–Moriya (DM) or Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) exchange factors, for different initial states. The results allowed to determine the range of parameters for which the analyzed system is characterized by the most favorable entanglement evolution.

- Piotr Kopszak: *On the representation theory of algebra of partially transposed permutation operators*

The algebra of permutation operators acting on n-fold tensor product of d-dimensional complex linear spaces that admit a deformation in form of partial transposition on the last k subsystems turn out to appear in various problems in physics, in quantum information theory in particular. This talk will cover the construction of matrix representations of such algebra (which are in fact matrix irreducible representations of Walled Brauer Algebra), together with the description of so-called partially reduced irreducible representations. An application to the study of performance of (multi) port-based teleportation protocol shall be provided as well.

- Ryszard Kostecki: Postquantum Brègman entropic projections and quasi-nonexpansive operators: general structure and some resource-theoretic applications

The Brègman family of relative entropies satisfies very nice geometric properties: 1) the generalised cosine and pythagorean theorems (i.e. an additive decomposition of relative entropy into parts corresponding to "signal" and "noise", with respect to an entropic projection, defined as a nonlinear map into a state which is a unique maximiser of relative entropy among a given closed convex subset); 2) their third order Taylor expansion determines a family of hessian (a.k.a. dually flat) geometries. Furthermore, in the reflexive Banach space setting, these relative entropies determine monoids of nonlinear quasi-nonexpansive operators, which satisfy a topological version of a generalised pythagorean theorem, and provide a generalisation of entropic projections. However, so far, an extension of a theory of Brègman relative entropies to arbitrary (nonreflexive, arbitrary dimensional) state spaces (in probabilistic, quantum, Jordan-algebraic, and base normed settings) was missing. We construct a solution to this problem, using a fusion of infinite-dimensional nonlinear convex analysis and nonlinear homeomorphy of Banach spaces, which allows us to provide a wide range of operator-algebraic examples (including, e.g., families of Brègman entropies and corresponding operators determined by the norm geometric characteristics of: noncommutative Orlicz spaces, nonassociative L_p spaces, generalised spin factors). The resulting theory is parallel to (and independent of) the theory of Csiszár–Morimoto/Kosaki–Petz f-divergences (with the Kullback–Leibler/Umegaki–Araki relative entropies belonging to an intersection of these two geometric worlds). As an application, we introduce categories and monoids of nonlinear (post)quantum quasi-nonexpansive operators with convergence controlled in terms of Brègman relative entropy, which give rise to a class of nonlinear (post)quantum resource theories. On the conceptual side, one can view these families of operators as a brègmanian analogue of CPTP maps, and as a generalisation of the method of inductive inference provided by the constrained relative entropy maximisation, with generalised pythagorean theorem playing the role of the fundamental geometric characteristic of an inference process.

- Gniewomir Sarbicki: *Multipartite generalisation of realignment criterion*

By a proper generalisation of the trace norm to hipermatrices we are able to formulate the realignment criterion in a multipartite scenario.

- Giovanni Scala: *Generalizing Choi map in M_3 beyond circulant scenario*

We introduce a family of positive linear maps in the algebra of 3×3 complex matrices, which generalizes the seminal positive non-decomposable map originally proposed by Choi. Necessary and sufficient conditions for decomposability are derived and demonstrated. The proposed maps offer a new method for the analysis of bound entangled states of two qutrits.

- Anna Szczepanek: *Invariant Measures for Quantum Trajectories and Dark Subspaces*

Quantum trajectories are Markov chains modeling the evolution of a quantum system subject to repeated indirect measurements. The renowned purification theorem by Kümmerer and Maassen [1] says that, asymptotically, a quantum trajectory either reaches the set of pure states or it gets trapped in a set of 'dark subspaces', i.e., subspaces from which no information can be extracted, where it performs a random walk. It was shown in [2] that in the case of (irreducible) systems with no dark subspaces, so when purification is guaranteed, quantum trajectories admit a unique invariant probability measure on the set of pure states and the convergence towards this measure is geometric. This talk concerns the case of systems that do have dark subspaces. By using max-likelihood estimations, we can show that the random walk between dark subspaces admits a unique invariant probability measure and that the convergence towards it is again geometric. We also discuss the problem of characterizing invariant measures for quantum trajectories inside dark subspaces, in particular we give some necessary conditions for uniqueness. Joint work with T. Benoist and C. Pellegrini.

[1] H. Maassen and B. Kümmerer, Purification of quantum trajectories, Dynamics & Stochastics, Lecture Notes-Monograph Series 48 (2006) [2] T. Benoist, M. Fraas, Y. Pautrat, and C. Pellegrini, Invariant measure for quantum trajectories, Prob. Theory and Related Fields 174 (2019).

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3 Posters:

- Michał Cholewiak: *Construction of general Sic-POVM attempt to prove Zauner conjecture*

Zauner conjecture states that Symmetric Informationally Complete Positive Operator Valued Measures (SIC-POVM) exist for all dimensions d of Hilbert space. Those measures can be represented as sets of rank one projectors. Vectors corresponding to that projectors constitute a d2-element set in d-dimensional Hilbert space. Those vectors form an equiangular set, which means that the square of modulus of inner product between every pair of vectors is constant and equal to

1/(d+1). Zauner conjecture has been investigated by many approaches, none of which was completely successful. We present the idea of weakening assumption for operators being rank one. In such a way, one can find the so-called General SIC-POVM. We present some conditions, which such a set must fulfill to become a true SIC-POVM.

- Paweł Cieśliński: The fastest generation of multipartite entanglement with natural interactions

Natural interactions among multiple quantum objects are fundamentally composed of two-body terms only. In contradistinction, single global unitaries that generate highly entangled states usually arise from Hamiltonians that couple multiple individual subsystems simultaneously. Is it then possible to create strongly nonclassical multipartite correlations with a single unitary generated by the natural interactions? How does the two-body restriction influence the time required for that? We answer these questions by studying the unitary evolutions of quantum states that are generated by Hamiltonians with two-body interactions. Our main focus is on the fastest generation of multipartite entangled Greenberger-Horne-Zeilinger (GHZ), W, Dicke and absolutely maximally entangled (AME) states for up to seven qubits. These results are obtained by constraining the energy in the system and accordingly can be seen as state-dependent quantum speed limits for natural interactions. They give rise to a counter-intuitive effect where the creation of particular entangled states with an increasing number of particles does not require more time. The methods used rely on extensive numerical simulations and analytical estimations.

- Agnieszka Martens: *Quantized Kepler-Coulomb dynamical models on two-dimensional constant curvature spaces.*

The paper is continuation of [1] where we have discussed some classical and quantization problems of rigid bodies of infinitesimal size moving in Riemannian spaces. Strictly speaking, we have considered oscillatory dynamical models on sphere and pseudosphere. Here we concentrate on Kepler-Coulomb potential models. We have used formulated in [1] the two-dimensional situation on the quantum level. The Sommerfeld polynomial method is used to perform the quantization of such problems.

[1] A. Martens: Rep. Math. Phys. 71, 381 (2013).

- Tomasz Młynik: Class of one-parameter k-positive maps in matrix algebras

Yang et al. [1] showed that each 2-positive map acting from $\mathcal{M}_3(C)$ into itself is decomposable. It is equivalent to the statement that each PPT state on $C^3 \otimes C^3$ has Schmidt number at most 2. It is a generalization of Perez-Horodecki criterion which states that each PPT state on $C^2 \otimes C^2$ or $C^2 \otimes C^3$ has Schmidt rank 1 *i.e.* is separable. Natural question arises whether the result of Yang at al. stays true for PPT states on $C^3 \otimes C^4$. This question can be considered also in higher dimensions. Motivated by these results and questions we construct a family of positive maps between matrix algebras of different dimensions depending on a parameter with the property that their k-positivity can be easily controlled. We found that in case where dimensions are differ by one we can give explicit analytic formula for parameter that guarantee k-positivity.

[1] Y. Yang, D. H. Leung and W. -S. Tang, All 2-positive linear maps from $M_3(C)$ to $M_3(C)$ are decomposable, Linear Alg. Appl. 503:233247, (2016).